5 Abstract coordinate systems

5.1 Introduction

An abstract coordinate system is a means of identifying positions in position-space by coordinate n-tuples. An abstract coordinate system is completely defined in terms of the mathematical structure of position-space. In this International Standard the term “coordinate system”, if not otherwise qualified, is defined to mean “abstract coordinate system.” Each coordinate system has a coordinate system type (see 5.4). Other coordinate system related concepts defined in this clause include coordinate-component surfaces and coordinate-component curves (see 5.5), linearity and other properties (see 5.6), and localization (see 5.7). Map projections and augmented map projections are defined and treated as special cases of the general abstract coordinate system concept (see 5.8). Standardized abstract coordinate systems are specified in 5.9.

In Clause 6 a temporal coordinate system is defined as a means of identifying events in the time continuum by coordinate 1-tuples using an abstract coordinate system of coordinate system type 1D. In Clause 8 a spatial coordinate system is defined as an abstract coordinate system suitably combined with a normal embedding (see Clause 7) as a means of identifying points in object-space by coordinate n-tuples.

5.2 Preliminaries

This International Standard takes a functional approach to the construction of coordinate systems. Annex A provides a concise summary of mathematical concepts and specifies the notational conventions used in this International Standard. In particular, Annex A defines the terms interior, one-to-one, smooth, smooth surface, smooth curve, orientation-preserving, and connected. The concept of $\mathbf{R}^n$ as a vector space, the point-set topology of $\mathbf{R}^n$, and the theory of real-valued functions on $\mathbf{R}^n$ are all assumed. Algebraic and analytic geometry, including the concepts of point, line, and plane, are also assumed. Together with such common concepts, a newly introduced concept replete will be used. A set D is replete if all points in D belong to the closure of the interior of D (see Annex A). A replete set is a generalization of an open set that allows the inclusion of boundary points. Boundary points are important in the definitions of certain coordinate systems.

5.3 Abstract CS

An *abstract Coordinate System* (CS) is a means of identifying a set of positions in an abstract Euclidean space that shall be comprised of:

a) a CS domain,

b) a generating function, and

c) a CS range,

where:

a) The *CS domain* shall be a connected replete domain in the Euclidean space of n-tuples ($1 \le n \le m$), called the *coordinate-space.*

b) The *generating function* shall be a one-to-one, smooth, orientation-preserving function from the CS domain onto the CS range.

c) The *CS range* shall be a set of positions in a Euclidean space of dimension m ($n \le m \le 3$), called the *position-space.* When $n = 2$ and $m = 3$, the CS range shall be a subset of a smooth surface[5]. When $n = 1$ and $m = 2$ or 3, the CS range shall be a subset of an implicitly specified smooth curve[6].

An element of the CS domain shall be called a *coordinate*[7]. The k^{th}-component of a coordinate n-tuple ($1 \le k \le n$) may be called the k^{th} *coordinate-component*. *Coordinate-component*[8] is the collective term for any k^{th} coordinate-component.

An element of the CS range shall be called a *position.* The *coordinate of a position* $\boldsymbol{p}$ shall be the unique coordinate whose generating function value is $\boldsymbol{p}$.

The generating function may be parameterized. The generating function parameters (if any) shall be called the *CS parameters.*

The inverse of the generating function shall be called the *inverse generating function*. The inverse generating function is one-to-one and is smooth and orientation-preserving in the interior of its domain, except at points in the image of the CS domain boundary points where it may be discontinuous. A CS may equivalently be defined by specifying the inverse generating function when the CS domain is an open set.

NOTE 1 The generating function of a CS is often specified by an algebraic and/or trigonometric description of a geometric relationship (see 5.3 Example). There are also CSs that do not have geometric derivations. The Mercator map projection (see Table 5.18) is specified to satisfy a functional requirement of conformality (see 5.8.3.2) rather than by a geometric construction.

EXAMPLE Polar CS: Considering the polar geometry depicted in Figure 5.1, define a generating function $\boldsymbol{F}$ as:

$$\boldsymbol{F}(\rho,\ \theta) = (x,\ y)$$

where:

$$x = \rho\cos(\theta), \text{ and } y = \rho\sin(\theta).$$

The CS domain of $\boldsymbol{F}$ in coordinate-space is $\{(\rho,\theta) \text{ in } \mathbf{R}^2 \mid 0 < \rho,\ 0 \le \theta < 2\pi\} \cup \{(0,0)\}$.
The CS range of $\boldsymbol{F}$ in position-space is $\mathbf{R}^2$.

This generating function is illustrated in Figure 5.2. The grey boxes with lighter grey edges in this figure represent the fact that the range in position-space extends indefinitely, and that the domain in coordinate-space extends indefinitely along the ρ-axis. The dotted grey edges indicate an open boundary. This CS range, CS domain, and generating function define an abstract CS representing polar coordinates as defined in mathematics [EDM, “Coordinates”]. The normative definition of the polar CS may be found in Table 5.33.

[5] The generating function properties and the implicit function theorem together imply that for each point in the interior of the CS domain, there is an open neighbourhood of the point whose image under the generating function lies in a smooth surface. This requirement specifies that there exists one smooth surface for all of the points in the CS domain. This requirement is specified to exclude mathematically pathological cases.

[6] The generating function properties and the implicit function theorem together imply that for each point in the interior of the CS domain, there is an open neighbourhood of the point whose image under the generating function lies in a smooth curve. This requirement specifies that there exists one implicitly-defined smooth curve for all the points in the CS domain. This requirement is specified to exclude mathematically pathological cases.

[7] The ISO 19111 term for this concept is “coordinate tuple”.

[8] The ISO 19111 term for this concept is “coordinate”.